

To Sabir Gusein-Zade, on the occasion of his 60th birthday

COMPLEX-ANALYTIC STRUCTURES ON MOMENT-ANGLE MANIFOLDS

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ABSTRACT. We show that the moment-angle manifolds corresponding to complete simplicial fans admit non-Kähler complex-analytic structures. This generalises the known construction of complex-analytic structures on polytopal moment-angle manifolds, coming from identifying them as LVM-manifolds. We proceed by describing Dolbeault cohomology and some Hodge numbers of moment-angle manifolds by applying the Borel spectral sequence to holomorphic principal bundles over toric varieties.

1. INTRODUCTION

Moment-angle complexes $\mathcal{Z}_{\mathcal{K}}$ are spaces acted on by a torus and parametrised by finite simplicial complexes \mathcal{K} . They are central objects in toric topology, and currently are gaining much interest in the homotopy theory. Due to their combinatorial origins, moment-angle complexes also find applications in combinatorial geometry and commutative algebra.

The construction of the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ ascends to the work of Davis–Januszkiewicz [DJ] on (quasi)toric manifolds; later $\mathcal{Z}_{\mathcal{K}}$ was described in [BP1] as a certain complex build up from polydiscs and tori. In the case when \mathcal{K} is a triangulation (simplicial subdivision) of a sphere, $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, referred to as the *moment-angle manifold*. Dual triangulations of simple convex polytopes P provide an important subclass of sphere triangulations; the corresponding *polytopal* moment-angle manifolds are known to be smooth [BP2, Lemma 6.2] and denoted \mathcal{Z}_P . The manifolds \mathcal{Z}_P corresponding to *Delzant polytopes* P are closely related to the construction of *Hamiltonian toric manifolds* via symplectic reduction; \mathcal{Z}_P arises as the level set for an appropriate moment map and therefore embeds into \mathbb{C}^m as a nondegenerate intersection of real quadrics with rational coefficients [BP2]. The topology of moment-angle complexes and manifolds is quite complicated even for small \mathcal{K} and P . The cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ was described in [BP1, Th. 4.2, 4.6] (see also [Pa, §4]), and explicit homotopy and diffeomorphism types for certain particular families of $\mathcal{Z}_{\mathcal{K}}$ and \mathcal{Z}_P were described in [GT] and [GL] respectively.

On the other hand, manifolds obtained as intersections of quadrics appeared in holomorphic dynamics as the spaces of leaves for holomorphic foliations in \mathbb{C}^m . Their topology was studied in [Lo]; this study led to a discovery of a new class of compact non-Kähler complex-analytic manifolds in the work of Lopez de Medrano and Verjovsky [LV] and Meersseman [Me], now known as the *LVM-manifolds*. Bosio and Meersseman were first to observe that the smooth manifolds underlying a large class of LVM-manifolds are exactly polytopal moment-angle manifolds, cf. [BM].

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It therefore became clear that the moment-angle manifolds \mathcal{Z}_P admit non-Kähler complex-analytic (or shortly complex) structures generalising the known families of Hopf and Calabi–Eckmann manifolds.

The aim of this paper is twofold. First, we intended to give an explicit intrinsic construction of complex structures on manifolds \mathcal{Z}_P within the theory of moment-angle complexes, without referring to the much developed theory of LVM-manifolds (although of course using ideas and methodology of this theory). Second, we aimed at generalising the construction of complex structures on \mathcal{Z}_P to the nonpolytopal case (this question was also raised in [BM, §15]), and extending our knowledge of invariants of these structures, such as Dolbeault cohomology and Hodge numbers.

In Section 2 we review the construction of moment-angle complexes and related spaces, such as the complement $U(\mathcal{K})$ of the coordinate subspace arrangement in \mathbb{C}^m corresponding to \mathcal{K} , where m is the number of vertices of \mathcal{K} . We then restrict attention to the case when \mathcal{K} is the underlying complex of a complete simplicial (but not necessarily rational) fan Σ in \mathbb{R}^n . Such \mathcal{K} are also known as *starshaped spheres*. We show that the corresponding moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ admits a smooth structure as the transverse space to the orbits of an action of \mathbb{R}^{m-n} on $U(\mathcal{K})$. The situation here is similar to the polytopal case of \mathcal{Z}_P , however, we do not have an explicit presentation of $\mathcal{Z}_{\mathcal{K}}$ as an intersection of quadrics. This also leaves open a question of whether moment-angle complexes $\mathcal{Z}_{\mathcal{K}}$ corresponding to more general sphere triangulations \mathcal{K} can be smoothed.

In Section 3 we modify the construction of Section 2 by replacing the action of \mathbb{R}^{m-n} on $U(\mathcal{K})$ by a holomorphic action of a complex group C isomorphic to \mathbb{C}^{ℓ} where $m - n = 2\ell$, provided that $m - n$ is even (this can always be achieved by adding a ‘ghost’ vertex to \mathcal{K}). The identification of $\mathcal{Z}_{\mathcal{K}}$ with the quotient $U(\mathcal{K})/C$ endows it with a structure of a complex manifold of dimension $m - \ell$.¹

In Section 4 we restrict to the polytopal case and relate the complex structure on \mathcal{Z}_P coming from the normal fan of P via the construction of Section 3 to the complex structure coming from identifying \mathcal{Z}_P as an LVM-manifold [BM]. We also relate the quadratic description of \mathcal{Z}_P to the quadratic description of LVM-manifolds.

In Section 5 we consider rational fans Σ , which give rise to toric varieties X_{Σ} . The Cox construction identifies X_{Σ} with the quotient of $U(\mathcal{K})$ by an action of an algebraic group G of dimension $m - n = 2\ell$. In the case of nonsingular X_{Σ} the inclusion of the complex ℓ -dimensional group C into the algebraic group G gives rise to a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}} \rightarrow X_{\Sigma}$ whose fibre is a compact complex torus of dimension ℓ . This extends the construction of [MV] of holomorphic principal bundles over projective toric varieties to the nonpolytopal (and therefore non projective) case.² An application of the Borel spectral sequence to the holomorphic bundle $\mathcal{Z}_{\mathcal{K}} \rightarrow X_{\Sigma}$ allows us to describe the Dolbeault cohomology groups of $\mathcal{Z}_{\mathcal{K}}$ in Theorem 5.4. From this description some Hodge numbers $h^{p,q}(\mathcal{Z}_{\mathcal{K}})$ may be calculated explicitly (Theorem 5.10).

¹While preparing this paper for publication we discovered that results similar to those of Section 3 were obtained by Tambour [Ta] using a different approach. In particular, Tambour constructed complex structures on manifolds $\mathcal{Z}_{\mathcal{K}}$ coming from *rationally* starshaped spheres \mathcal{K} , by relating them to a class of generalised LVM-manifolds described by Bosio in [Bos].

²In view of the results of [Ta], this extension is covered by the results of Cupit-Foutou and Zaffran [CZ].

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2. MOMENT-ANGLE COMPLEXES AND MANIFOLDS

Let \mathcal{K} be an abstract simplicial complex on the set $[m] = \{1, \dots, m\}$, i.e. a collection of subsets $I = \{i_1, \dots, i_k\} \subset [m]$ closed under inclusion. We refer to $I \in \mathcal{K}$ as *simplices* and always assume that $\emptyset \in \mathcal{K}$. We denote by $|\mathcal{K}|$ a geometric realisation of \mathcal{K} , which is a topological space.

Consider the unit polydisc in \mathbb{C}^m ,

$$\mathbb{D}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \quad i = 1, \dots, m\}.$$

Given $I \subset [m]$, define

$$B_I := \{(z_1, \dots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ for } j \notin I\},$$

Following [BP2], define the *moment-angle complex* $\mathcal{Z}_{\mathcal{K}}$ as

$$(2.1) \quad \mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| = 1, \quad i = 1, \dots, m\}.$$

The definition of the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is a particular case of the following general construction.

Construction 2.1 (\mathcal{K} -power). Let X be a space, and W a subspace of X . Given $I \subset [m]$, set

$$(2.2) \quad (X, W)^I := \{(x_1, \dots, x_m) \in X^m : x_j \in W \text{ for } j \notin I\} \cong \prod_{i \in I} X \times \prod_{i \notin I} W,$$

and define the \mathcal{K} -power (also known as the *polyhedral product*) of (X, W) as

$$(X, W)^{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} (X, W)^I \subset X^m.$$

Note that we do not assume that \mathcal{K} contains all one-element subsets $\{i\} \subset [m]$; we refer to $\{i\} \notin \mathcal{K}$ as a *ghost vertex*. Obviously, if \mathcal{K} has k ghost vertices, then $(X, W)^{\mathcal{K}} \cong (X, W)^{\mathcal{K}'} \times W^k$, where \mathcal{K}' does not have ghost vertices.

It follows from the definition that

- $\mathcal{Z}_{\mathcal{K}} = (\mathbb{D}, \mathbb{T})^{\mathcal{K}}$, where \mathbb{T} is the unit circle.

Other important particular cases of \mathcal{K} -powers include;

- the standard cubical decomposition of the quotient $\mathcal{Z}_{\mathcal{K}}/\mathbb{T}^m \cong \text{cone}|\mathcal{K}|$ (see [BP2, §6.2]), namely,

$$(2.3) \quad \text{cc}(\mathcal{K}) := (\mathbb{I}, 1)^{\mathcal{K}} \subset \mathbb{I}^m,$$

where \mathbb{I} is the unit segment $[0, 1]$ and \mathbb{I}^m is the unit m -cube;

- the *complex coordinate subspace arrangement complement* corresponding to \mathcal{K} (see [BP2, §8.2]):

$$U(\mathcal{K}) := \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}},$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

We obviously have $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$. Moreover, $\mathcal{Z}_{\mathcal{K}}$ is a \mathbb{T}^m -equivariant deformation retract of $U(\mathcal{K})$ for every \mathcal{K} [BP2, Th. 8.9]. For more generalisations and applications of \mathcal{K} -powers see [BBCG].

It is shown in [BP2, Lemma 6.13] that $\mathcal{Z}_{\mathcal{K}}$ is a (closed) topological manifold whenever $|\mathcal{K}|$ is a triangulation of a sphere; in this case we refer to $\mathcal{Z}_{\mathcal{K}}$ as a *moment-angle manifold*. If $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope P , then $\mathcal{Z}_P := \mathcal{Z}_{\mathcal{K}_P}$ can be canonically smoothed [BP2, Lemma 6.2] using the standard structure of manifold with corners on P .

The aim of this section is to show that moment-angle manifolds $\mathcal{Z}_{\mathcal{K}}$ admit smooth structures for a wider class of simplicial complexes \mathcal{K} , namely, those corresponding to *complete simplicial fans*.

A set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^n$ defines a convex polyhedral *cone*

$$\sigma = \{\mu_1 \mathbf{a}_1 + \dots + \mu_k \mathbf{a}_k : \mu_i \in \mathbb{R}, \mu_i \geq 0\}.$$

A cone is *rational* if its generating vectors can be chosen from the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, and is *strongly convex* if it does not contain a line. A cone is *simplicial* (respectively, *regular*) if it is generated by a part of basis of \mathbb{R}^n (respectively, \mathbb{Z}^n).

A *fan* is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones in some \mathbb{R}^n such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is *rational* (respectively, *simplicial*, *regular*) if every cone in Σ is rational (respectively, simplicial, regular). A fan $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is called *complete* if $\sigma_1 \cup \dots \cup \sigma_s = \mathbb{R}^n$.

In this section we assume that Σ is a simplicial (but not necessarily rational) fan in \mathbb{R}^n with m one-dimensional cones, for which we choose generator vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$. Define the *underlying simplicial complex* \mathcal{K}_Σ on $[m]$ as the collection of subsets $I \subset [m]$ such that $\{\mathbf{a}_i : i \in I\}$ spans a cone of Σ . Note that Σ is complete if and only if $|\mathcal{K}_\Sigma|$ is a triangulation of S^{n-1} . Now consider the linear map

$$(2.4) \quad \Lambda_{\mathbb{R}} : \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i,$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ is the standard basis of \mathbb{R}^m . Let

$$\mathbb{R}_{>}^m = \{(y_1, \dots, y_m) \in \mathbb{R}^m : y_i > 0\}$$

be the multiplicative group of m -tuples of positive real numbers, and define

$$(2.5) \quad R_\Sigma := \exp(\text{Ker } \Lambda_{\mathbb{R}}) = \{(y_1, \dots, y_m) \in \mathbb{R}_{>}^m : \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u} \rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^n . Note that $R_\Sigma \cong \mathbb{R}_{>}^{m-n}$ if Σ is complete (or contains at least one n -dimensional cone).

We let $\mathbb{R}_{>}^m$ act on the complement $U(\mathcal{K}_\Sigma)$ by coordinatewise multiplications and consider the restricted action of the subgroup $R_\Sigma \subset \mathbb{R}_{>}^m$. Recall that an action of

a topological group G on a space X is *proper* if the map $h: G \times X \rightarrow X \times X$, $(g, x) \mapsto (gx, x)$ is proper.

Theorem 2.2. *Let Σ be a complete simplicial fan in \mathbb{R}^n with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Then*

- (a) *the group R_Σ acts on $U(\mathcal{K})$ freely and properly, and the quotient $U(\mathcal{K})/R_\Sigma$ is a smooth $(m+n)$ -dimensional manifold;*
- (b) *$U(\mathcal{K})/R_\Sigma$ is \mathbb{T}^m -equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.*

Therefore, $\mathcal{Z}_\mathcal{K}$ can be smoothed.

Proof. We first prove statement (a). The fact that R_Σ acts on $U(\mathcal{K})$ freely is standard, and is proved in the same way as [BP2, Prop. 6.5]. Indeed, a point $\mathbf{z} \in U(\mathcal{K})$ has a nontrivial isotropy subgroup with respect to the action of $\mathbb{R}_>^m$ only if some of its coordinates vanish. These $\mathbb{R}_>^m$ -isotropy subgroups are of the form $(\mathbb{R}_>, 1)^I$, see (2.2), for some $I \in \mathcal{K}$. The restriction of $\exp \Lambda_\mathbb{R}$ to every such $(\mathbb{R}_>, 1)^I$ is an injection. Therefore, $R_\Sigma = \exp(\text{Ker } \Lambda_\mathbb{R})$ intersects every $\mathbb{R}_>^m$ -isotropy subgroup only at the unit, which implies that the R_Σ -action on $U(\mathcal{K})$ is free.

Let us prove that the R_Σ -action on $U(\mathcal{K})$ is proper. Let $\{\mathbf{g}^i\} \in R_\Sigma$, $\{\mathbf{x}^i\} \in U(\mathcal{K})$ be sequences of points such that $\{\mathbf{x}^i\}$ and $\{\mathbf{y}^i\} := \{\mathbf{g}^i \mathbf{x}^i\}$ have limits in $U(\mathcal{K})$:

$$\{\mathbf{x}^i\} \rightarrow \mathbf{x} = (x_1, \dots, x_m), \quad \{\mathbf{y}^i\} \rightarrow \mathbf{y} = (y_1, \dots, y_m).$$

We claim that a subsequence of $\{\mathbf{g}^i\}$ has limit in R_Σ . Indeed, every \mathbf{g}^i is represented by an m -tuple,

$$\mathbf{g}^i = (\exp \gamma_1^i, \dots, \exp \gamma_m^i) \in \exp \mathbb{R}^m.$$

Passing to a subsequence of $\{\mathbf{g}^i\}$ if necessary, we may assume that every sequence $\{\gamma_k^i\}$, $k = 1, \dots, m$, has a finite or infinite limit (including $\pm\infty$). Let

$$I_+ = \{k: \gamma_k^i \rightarrow +\infty\} \subset [m], \quad I_- = \{k: \gamma_k^i \rightarrow -\infty\} \subset [m].$$

Since sequences $\{\mathbf{x}^i\}$ and $\{\mathbf{y}^i\}$ are bounded, $x_k = 0$ for $k \in I_+$ and $y_k = 0$ for $k \in I_-$. The definition of $U(\mathcal{K})$ implies that I_+ and I_- are simplices of \mathcal{K} . Let σ_+, σ_- be the corresponding cones of the fan Σ . Since $\sigma_+ \cap \sigma_- = \{0\}$, there exists a linear function ξ on \mathbb{R}^n such that $\xi(v) > 0$ for $v \in \sigma_+, v \neq 0$, and $\xi(v) < 0$ for $v \in \sigma_-, v \neq 0$. Recall that $\mathbf{g}^i \in R_\Sigma = \exp(\text{Ker } \Lambda_\mathbb{R})$, therefore,

$$0 = \xi \left(\sum_{k=1}^m \gamma_k^i \mathbf{a}_k \right) = \sum_{k=1}^m \gamma_k^i \xi(\mathbf{a}_k).$$

Hence, both I_+ and I_- are empty (otherwise the latter sum tends to infinity). Thus, \mathbf{g}^i converges to a point in R_Σ , and the preimage of any compact subspace in $U(\mathcal{K}) \times U(\mathcal{K})$ under the action map $h: R_\Sigma \times U(\mathcal{K}) \rightarrow U(\mathcal{K}) \times U(\mathcal{K})$ is compact. This proves the properness of the action. Since the Lie group $R(\Sigma)$ acts smoothly, freely and properly on $U(\mathcal{K})$, the orbit space $U(\mathcal{K})/R(\Sigma)$ admits a structure of a smooth manifold by the standard result [Le, Th. 9.16].

In our case it is possible to construct a smooth atlas on $U(\mathcal{K})/R_\Sigma$ explicitly. To do this, it is convenient to pre-factorise everything by the action of \mathbb{T}^m . The quotient $U(\mathcal{K})/\mathbb{T}^m$ has the following decomposition as a \mathcal{K} -power:

$$U(\mathcal{K})/\mathbb{T}^m = (\mathbb{R}_{\geq}, \mathbb{R}_{>})^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}_{\geq}, \mathbb{R}_{>})^I,$$

where \mathbb{R}_{\geq} is the set of nonnegative reals. Since the fan Σ is complete, we may take the union above only over n -element simplices $I = \{i_1, \dots, i_n\} \in \mathcal{K}$. Consider one

such simplex I ; the generators of the corresponding n -dimensional cone $\sigma \in \Sigma$ are $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}$. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ denote the dual basis of \mathbb{R}^n (which is a generator set of the dual cone σ^*). Then we have $\langle \mathbf{a}_{i_k}, \mathbf{u}_j \rangle = \delta_{jk}$. Now consider the map

$$p_I: (\mathbb{R}_{\geq}, \mathbb{R}_{>})^I \rightarrow \mathbb{R}_{\geq}^n$$

$$(y_1, \dots, y_m) \mapsto \left(\prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}_1 \rangle}, \dots, \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}_n \rangle} \right),$$

where we set $0^0 = 1$. Note that zero cannot occur with a negative exponent in the right hand side, hence p_I is well defined as a continuous map. Every $(\mathbb{R}_{\geq}, \mathbb{R}_{>})^I$ is R_Σ -invariant, and it follows from (2.5) that p_I induces an injective map

$$q_I: (\mathbb{R}_{\geq}, \mathbb{R}_{>})^I / R_\Sigma \rightarrow \mathbb{R}_{\geq}^n.$$

This map is also surjective since every $(x_1, \dots, x_n) \in \mathbb{R}_{\geq}^n$ is covered by (y_1, \dots, y_m) where $y_{i_j} = x_j$ for $1 \leq j \leq n$ and $y_k = 1$ for $k \notin \{i_1, \dots, i_n\}$. Hence, q_I is a homeomorphism. It is covered by a \mathbb{T}^m -equivariant homeomorphism

$$\bar{q}_I: (\mathbb{C}, \mathbb{C}^\times)^I / R_\Sigma \rightarrow \mathbb{C}^n \times \mathbb{T}^{m-n},$$

where \mathbb{C}^n is identified with a quotient of $\mathbb{R}_{\geq}^n \times \mathbb{T}^n$ in the standard way (e.g., using the polar coordinates in each factor). Since $U(\mathcal{K})/R_\Sigma$ is covered by open subsets $(\mathbb{C}, \mathbb{C}^\times)^I / R_\Sigma$, and $\mathbb{C}^n \times \mathbb{T}^{m-n}$ embeds as an open subset in \mathbb{R}^{m+n} , the set of homeomorphisms $\{\bar{q}_I: I \in \mathcal{K}\}$ provides an atlas for $U(\mathcal{K})/R_\Sigma$ (compare [BP2, proof of Lemma 6.2]). The change of coordinates transformations $\bar{q}_J \bar{q}_I^{-1}: \mathbb{C}^n \times \mathbb{T}^{m-n} \rightarrow \mathbb{C}^n \times \mathbb{T}^{m-n}$ are smooth by inspection; thus $U(\mathcal{K})/R_\Sigma$ is a smooth manifold.

Remark. The set of homeomorphisms $\{q_I: (\mathbb{R}_{\geq}, \mathbb{R}_{>})^I / R_\Sigma \rightarrow \mathbb{R}_{\geq}^n\}$ defines a canonical atlas for the smooth manifold with corners $\mathcal{Z}_{\mathcal{K}}/\mathbb{T}^m$. If $\mathcal{K} = \mathcal{K}_P$ for a simple polytope P , then this smooth structure with corners coincides with that of P .

If X is a Hausdorff locally compact space with a proper G -action, and $Y \subset X$ a compact subspace which intersects every G -orbit at a single point, then Y is homeomorphic to the orbit space X/G . Therefore, to prove statement (b) it is enough to verify that every R_Σ -orbit intersects $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$ at a single point. We first prove that the R_Σ -orbit of any $\mathbf{y} \in U(\mathcal{K})/\mathbb{T}^m = (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}}$ intersects $\mathcal{Z}_{\mathcal{K}}/\mathbb{T}^m$ at a single point. For this we use cubical decomposition $\text{cc}(\mathcal{K})$ of $\mathcal{Z}_{\mathcal{K}}/\mathbb{T}^m$, see (2.3).

Assume first that $\mathbf{y} \in \mathbb{R}_{>}^m$. The R_Σ -action on $\mathbb{R}_{>}^m$ is obtained by exponentiating the linear action of $\text{Ker } \Lambda_{\mathbb{R}}$ on \mathbb{R}^m . Consider the subset $(\mathbb{R}_{\leq}, 0)^{\mathcal{K}} \subset \mathbb{R}^m$, where \mathbb{R}_{\leq} denotes the set of nonpositive reals. It is taken by the exponential map $\exp: \mathbb{R}^m \rightarrow \mathbb{R}_{>}^m$ homeomorphically onto $((0, 1], 1)^{\mathcal{K}} = \text{cc}^\circ(\mathcal{K}) \subset \mathbb{R}_{>}^m$, where the latter is the relative interior of $\text{cc}(\mathcal{K})$. The map

$$(2.6) \quad \Lambda_{\mathbb{R}}: (\mathbb{R}_{\leq}, 0)^{\mathcal{K}} \rightarrow \mathbb{R}^n$$

takes every $(\mathbb{R}_{\leq}, 0)^I$ to $-\sigma$, where $\sigma \in \Sigma$ is the cone corresponding to $I \in \mathcal{K}$. Since Σ is complete, map (2.6) is one-to-one.

The orbit of \mathbf{y} under the action of R_Σ coincides with the set of points $\mathbf{w} \in \mathbb{R}_{>}^m$ such that $\exp \Lambda_{\mathbb{R}} \mathbf{w} = \exp \Lambda_{\mathbb{R}} \mathbf{y}$. Since $\Lambda_{\mathbb{R}} \mathbf{y} \in \mathbb{R}^n$ and map (2.6) is one-to-one, there is a unique point $\mathbf{y}' \in (\mathbb{R}_{\leq}, 0)^{\mathcal{K}}$ such that $\Lambda_{\mathbb{R}} \mathbf{y}' = \Lambda_{\mathbb{R}} \mathbf{y}$. Since $\exp \Lambda_{\mathbb{R}} \mathbf{y}' \subset \text{cc}^\circ(\mathcal{K})$, the R_Σ -orbit of \mathbf{y} intersects $\text{cc}^\circ(\mathcal{K})$ and therefore $\text{cc}(\mathcal{K})$ at a unique point.

Now let $\mathbf{y} \in (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}}$ be an arbitrary point. Let $I(\mathbf{y}) \in \mathcal{K}$ be the set of zero coordinates of \mathbf{y} , and let $\sigma \in \Sigma$ be the cone corresponding to $I(\mathbf{y})$. The cones

containing σ constitute a fan $\text{St } \sigma$ (called the *star* of σ) in the quotient space $\mathbb{R}^n / \mathbb{R}^{I(\mathbf{y})}$. Its underlying simplicial complex is the *link* $\text{lk } I(\mathbf{y})$ of $I(\mathbf{y})$ in \mathcal{K} . Now observe that the action of R_Σ on the set

$$\{(y_1, \dots, y_m) \in (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}} : y_i = 0 \text{ for } i \in I(\mathbf{y})\} \cong (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\text{lk } I(\mathbf{y})}$$

coincides with the action of the group $R_{\text{St } \sigma}$. Now we can repeat the above arguments for the complete fan $\text{St } \sigma$ and the action of $R_{\text{St } \sigma}$ on $(\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\text{lk } I(\mathbf{y})}$. As the result, we obtain that every R_Σ -orbit intersects $\text{cc}(\mathcal{K})$ at a unique point.

To finish the proof of (b) we consider the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}} & \longrightarrow & U(\mathcal{K}) \\ \downarrow & & \downarrow \pi \\ \text{cc}(\mathcal{K}) & \longrightarrow & (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}} \end{array}$$

where the horizontal arrows are embeddings and the vertical ones are projections onto the quotients of \mathbb{T}^m -actions. Note that the projection π commutes with the R_Σ -actions on $U(\mathcal{K})$ and $(\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}}$, and the subgroups R_Σ and \mathbb{T}^m of $(\mathbb{C}^\times)^m$ intersect trivially. It follows that every R_Σ -orbit intersects the full preimage $\pi^{-1}(\text{cc}(\mathcal{K})) = \mathcal{Z}_{\mathcal{K}}$ at a unique point. Indeed, assume that \mathbf{z} and $r\mathbf{z}$ are in $\mathcal{Z}_{\mathcal{K}}$ for some $\mathbf{z} \in U(\mathcal{K})$ and $r \in R_\Sigma$. Then $\pi(\mathbf{z})$ and $\pi(r\mathbf{z}) = r\pi(\mathbf{z})$ are in $\text{cc}(\mathcal{K})$, which implies that $\pi(\mathbf{z}) = \pi(r\mathbf{z})$. Hence, $\mathbf{z} = t r \mathbf{z}$ for some $t \in \mathbb{T}^m$. We may assume that $\mathbf{z} \in (\mathbb{C}^\times)^m$, so that the action of both R_Σ and \mathbb{T}^m is free (otherwise consider the action on $U(\text{lk } I(\mathbf{z}))$ where $I(\mathbf{z}) \in \mathcal{K}$ is the set of zero coordinates of \mathbf{z}). It follows that $t r = \mathbf{1}$, which implies that $r = \mathbf{1}$, since R_Σ and \mathbb{T}^m intersect trivially. \square

Remark. Our construction of a smooth structure on $\mathcal{Z}_{\mathcal{K}_\Sigma}$ depends on the geometry of Σ . However, we expect that the smooth structures coming from fans Σ and Σ' are the same whenever the underlying simplicial complexes \mathcal{K}_Σ and $\mathcal{K}_{\Sigma'}$ are isomorphic. Equivalently, the quotients $\mathcal{Z}_{\mathcal{K}_\Sigma} / \mathbb{T}^m$ and $\mathcal{Z}_{\mathcal{K}_{\Sigma'}} / \mathbb{T}^m$ are diffeomorphic as manifolds with corners whenever $\mathcal{K}_\Sigma = \mathcal{K}_{\Sigma'}$. It is true in the polytopal case (see also discussion in Section 4), and also for those fans Σ which are *shellable*. (A shelling order allows us to use an inductive argument, at each step extending a diffeomorphism between two $(k-1)$ -balls in the boundaries of k -balls to the whole k -balls.)

We do not know if Theorem 2.2 generalises to other sphere triangulations:

Question 2.3. *Describe the class of sphere triangulations \mathcal{K} for which the moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ admits a smooth structure.*

3. COMPLEX-ANALYTIC STRUCTURES

Here we show that the even-dimensional moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}_{>}^{m-n}$ on $U(\mathcal{K})$ (whose quotient is $\mathcal{Z}_{\mathcal{K}}$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

In this section we assume that $m-n$ is even. We can always achieve this by formally adding an ‘empty’ one-dimensional cone to Σ ; this corresponds to adding a ghost vertex to \mathcal{K} , or multiplying $\mathcal{Z}_{\mathcal{K}}$ by a circle. The column of matrix $\Lambda_{\mathbb{R}}$ corresponding to the ‘empty’ 1-cone is set to be zero. Set $\ell := \frac{m-n}{2}$.

We identify \mathbb{C}^m (as a real vector space) with \mathbb{R}^{2m} using the map

$$(3.1) \quad (z_1, \dots, z_m) \mapsto (x_1, y_1, \dots, x_m, y_m),$$

where $z_k = x_k + iy_k$ for $k = 1, \dots, m$, and denote the \mathbb{R} -linear map $\mathbb{C}^m \rightarrow \mathbb{R}^m$, $(z_1, \dots, z_m) \mapsto (x_1, \dots, x_m)$ by Re .

Construction 3.1. Choose a linear map $\Psi: \mathbb{C}^\ell \rightarrow \mathbb{C}^m$ satisfying two conditions:

- (a) $\text{Re} \circ \Psi: \mathbb{C}^\ell \rightarrow \mathbb{R}^m$ is a monomorphism.
- (b) $\Lambda_{\mathbb{R}} \circ \text{Re} \circ \Psi = 0$.

This corresponds to choosing a complex structure and specifying a complex basis in the real vector space $\text{Ker } \Lambda_{\mathbb{R}} \cong \mathbb{R}^{2\ell}$. Consider the following commutative diagram:

$$(3.2) \quad \begin{array}{ccccccc} \mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\text{Re}} & \mathbb{R}^m & \xrightarrow{\Lambda_{\mathbb{R}}} & \mathbb{R}^n \\ & & \downarrow \exp & & \downarrow \exp & & \downarrow \exp \\ & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}_{>}^m & \xrightarrow{\exp \Lambda_{\mathbb{R}}} & \mathbb{R}_{>}^n \end{array}$$

where the vertical arrows are the componentwise exponential maps, and $|\cdot|$ denotes the map $(z_1, \dots, z_m) \mapsto (|z_1|, \dots, |z_m|)$. Now set

$$(3.3) \quad C_{\Psi, \Sigma} := \exp \Psi(\mathbb{C}^\ell) = \{ (e^{\langle \psi_1, \mathbf{w} \rangle}, \dots, e^{\langle \psi_m, \mathbf{w} \rangle}) \in (\mathbb{C}^\times)^m \}$$

where $\mathbf{w} = (w_1, \dots, w_\ell) \in \mathbb{C}^\ell$, ψ_i denotes the i th row of the $m \times \ell$ -matrix $\Psi = (\psi_{ij})$, and $\langle \psi_i, \mathbf{w} \rangle = \psi_{i1}w_1 + \dots + \psi_{i\ell}w_\ell$. Then $C_{\Psi, \Sigma} \cong \mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Example 3.2. Let \mathcal{K} be a simplicial complex on a two element set consisting of the empty set only (that is, \mathcal{K} has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $\Lambda_{\mathbb{R}}: \mathbb{R}^2 \rightarrow 0$ is a zero map. Let $\Psi: \mathbb{C} \rightarrow \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that subgroup (3.3) is

$$C = \{(e^z, e^{\alpha z})\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) of Construction 3.1 is void, while condition (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp \Psi: \mathbb{C} \rightarrow (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T_{\mathbb{C}}^2$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T_{\mathbb{C}}^2(\alpha).$$

If we start with the empty simplicial complex on the set of 2ℓ elements (so that $n = 0$, $m = 2\ell$), we may obtain an arbitrary compact complex ℓ -dimensional torus $T_{\mathbb{C}}^{2\ell}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C_{\Psi, \Sigma}$, see [Me, Th. 1].

Theorem 3.3. *Let Σ be a complete simplicial fan in \mathbb{R}^n with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Assume that $m - n = 2\ell$ and let $C_{\Psi, \Sigma}$ be a subgroup of $(\mathbb{C}^\times)^m$ defined by (3.3). Then*

- (a) *the holomorphic action of the group $C_{\Psi, \Sigma}$ on $U(\mathcal{K})$ is free and proper, and the quotient $U(\mathcal{K})/C_{\Psi, \Sigma}$ has a structure of a compact complex manifold of complex dimension $m - \ell$;*
- (b) *there is a \mathbb{T}^m -equivariant diffeomorphism between $U(\mathcal{K})/C_{\Psi, \Sigma}$ and $\mathcal{Z}_{\mathcal{K}}$ defining a complex structure on $\mathcal{Z}_{\mathcal{K}}$, in which \mathbb{T}^m acts by holomorphic transformations.*

Proof. We first prove statement (a). The isotropy subgroups of the $(\mathbb{C}^\times)^m$ -action on $U(\mathcal{K})$ are of the form $(\mathbb{C}^\times, 1)^I$ for $I \in \mathcal{K}$. In order to show that the subgroup $C_{\Psi, \Sigma}$ acts freely we need to check that $C_{\Psi, \Sigma}$ intersects every $(\mathbb{C}^\times)^m$ -isotropy subgroup only at the unit. Since $C_{\Psi, \Sigma}$ embeds into $\mathbb{R}_{>}^m$ by (3.2), it enough to check that the

image of $C_{\Psi,\Sigma}$ in $\mathbb{R}_{>}^m$ intersects the image of $(\mathbb{C}^\times, 1)^I$ only at the unit. The former image is R_Σ and the latter image is $(\mathbb{R}_{>}, 1)^I$; the triviality of their intersection follows from Theorem 2.2 (a).

Now we prove the properness of this action. Consider the projection $\pi: U(\mathcal{K}) \rightarrow (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}}$ onto the quotient of the \mathbb{T}^m -action, and the commutative square

$$\begin{array}{ccc} C_{\Psi,\Sigma} \times U(\mathcal{K}) & \xrightarrow{h_{\mathbb{C}}} & U(\mathcal{K}) \times U(\mathcal{K}) \\ \downarrow i \times \pi & & \downarrow \pi \times \pi \\ R_\Sigma \times (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}} & \xrightarrow{h_{\mathbb{R}}} & (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}} \times (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}} \end{array},$$

where $h_{\mathbb{C}}$ and $h_{\mathbb{R}}$ denote the group action maps, and $i: C_{\Psi,\Sigma} \rightarrow R_\Sigma$ is the isomorphism given by the restriction of $|\cdot|: (\mathbb{C}^\times)^m \rightarrow \mathbb{R}_{>}^m$. The preimage $h_{\mathbb{C}}^{-1}(V)$ of any compact subset $V \in U(\mathcal{K}) \times U(\mathcal{K})$ is a closed subset in the set $W = (i \times \pi)^{-1} \circ h_{\mathbb{R}}^{-1} \circ (\pi \times \pi)(V)$. The image $\pi \times \pi(V)$ is compact, the action of R_Σ on $(\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}}$ is proper by the same argument as used in the proof of Theorem 2.2 (a), and the map $i \times \pi$ is proper, since it is the quotient projection of a compact group action. Hence, W is a compact subset in $C_{\Psi,\Sigma} \times U(\mathcal{K})$, and $h_{\mathbb{C}}^{-1}(V)$ is compact as a closed subset in W .

The complex group $C_{\Psi,\Sigma}$ acts holomorphically, freely and properly on the complex manifold $U(\mathcal{K})$, therefore by the complex analogue of [Le, Th. 7.10], the orbit space admits a structure of a complex manifold.

Like in the real situation of Section 2, it is possible to construct an atlas of $U(\mathcal{K})/C_{\Psi,\Sigma}$ explicitly. Since the action of $C_{\Psi,\Sigma}$ on the quotient $U(\mathcal{K})/\mathbb{T}^m = (\mathbb{R}_{\geq}, \mathbb{R}_{>})^{\mathcal{K}}$ coincides with the action of R_Σ on the same space, the quotient of $U(\mathcal{K})/C_{\Psi,\Sigma}$ by the action of \mathbb{T}^m has exactly the same structure of a smooth manifold with corners as the quotient of $U(\mathcal{K})/R_\Sigma$ by \mathbb{T}^m (see the proof of Theorem 2.2). This structure is determined by the atlas $\{q_I: (\mathbb{R}_{\geq}, \mathbb{R}_{>})^I/R_\Sigma \rightarrow \mathbb{R}_{\geq}^n\}$, which lifts to a covering of $U(\mathcal{K})/C_{\Psi,\Sigma}$ by open subsets $(\mathbb{C}, \mathbb{C}^\times)^I/C_{\Psi,\Sigma}$. For every $I \in \mathcal{K}$, the subset $(\mathbb{C}, \mathbb{T})^I \subset (\mathbb{C}, \mathbb{C}^\times)^I$ intersects any orbit of the $C_{\Psi,\Sigma}$ -action on $(\mathbb{C}, \mathbb{C}^\times)^I$ transversely at a single point. Therefore, every $(\mathbb{C}, \mathbb{C}^\times)^I/C_{\Psi,\Sigma} \cong (\mathbb{C}, \mathbb{T})^I$ acquires a structure of a complex manifold. Since $(\mathbb{C}, \mathbb{C}^\times)^I \cong \mathbb{C}^n \times (\mathbb{C}^\times)^{m-n}$, and the action of $C_{\Psi,\Sigma}$ on the $(\mathbb{C}^\times)^{m-n}$ factor is free, the complex manifold $(\mathbb{C}, \mathbb{C}^\times)^I/C_{\Psi,\Sigma}$ is the total space of a holomorphic \mathbb{C}^n -bundle over the compact complex torus $T_{\mathbb{C}}^{2\ell} = (\mathbb{C}^\times)^{m-n}/C_{\Psi,\Sigma}$ (see Example 3.2). Writing trivialisations of these \mathbb{C}^n -bundles for every I , we obtain a holomorphic atlas for $U(\mathcal{K})/C_{\Psi,\Sigma}$.

The proof of statement (b) follows the lines of the proof of Theorem 2.2 (b). We need to show that every $C_{\Psi,\Sigma}$ -orbit intersects $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$ at a single point. First we show that the $C_{\Psi,\Sigma}$ -orbit of every point in $U(\mathcal{K})/\mathbb{T}^m$ intersects $\mathcal{Z}_{\mathcal{K}}/\mathbb{T}^m = \text{cc}(\mathcal{K})$ at a single point; this follows from the fact that the actions of $C_{\Psi,\Sigma}$ and R_Σ coincide on $U(\mathcal{K})/\mathbb{T}^m$. Then we show that every $C_{\Psi,\Sigma}$ -orbit intersects the full preimage $\pi^{-1}(\text{cc}(\mathcal{K}))$ at a single point using the fact that $C_{\Psi,\Sigma}$ and \mathbb{T}^m have trivial intersection in $(\mathbb{C}^\times)^m$. \square

Remark. Unlike the smooth structure, the complex structure on $\mathcal{Z}_{\mathcal{K}}$ depends on both the geometry of Σ and the choice of Ψ in Construction 3.1. The fact that the choice of Ψ affects the complex structure is already clear from Example 3.2.

Example 3.4 (Hopf manifold). Let Σ be a complete simplicial fan in \mathbb{R}^n whose cones are generated by all proper subsets of the set of $n+1$ vectors $\mathbf{e}_1, \dots, \mathbf{e}_n, -\mathbf{e}_1 - \dots - \mathbf{e}_n$.

$\dots - e_n$. To make $m - n$ even we also add one ‘empty’ 1-cone. We therefore have $m = n + 2$, $\ell = 1$. Then $\Lambda_{\mathbb{R}}: \mathbb{R}^{n+2} \rightarrow \mathbb{R}^n$ is given by the matrix $(\mathbf{0} \ I \ -\mathbf{1})$, where I is the unit $n \times n$ matrix, and $\mathbf{0}, \mathbf{1}$ are the n -columns of zeros and units respectively.

We have that \mathcal{K} is the boundary of an n -dimensional simplex with $n + 1$ vertices and 1 ghost vertex, $\mathcal{Z}_{\mathcal{K}} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^{\times} \times (\mathbb{C}^{n+1} \setminus \{0\})$. Let $\Psi: \mathbb{C} \rightarrow \mathbb{C}^{n+2}$ be given by $z \mapsto (z, \alpha z, \dots, \alpha z)$ for some $\alpha \in \mathbb{C}$. Like in Example 3.2, conditions of Construction 3.1 imply that α is not a real number, and $\exp \Psi$ embeds \mathbb{C} as the following subgroup in $(\mathbb{C}^{\times})^{n+2}$:

$$C = \{(e^z, e^{\alpha z}, \dots, e^{\alpha z}) : z \in \mathbb{C}\} \subset (\mathbb{C}^{\times})^{n+2}.$$

By Theorem 3.3, $\mathcal{Z}_{\mathcal{K}}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^{\times} \times (\mathbb{C}^{n+1} \setminus \{0\}) / \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z} \mathbf{w})\} \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \{\mathbf{w} \sim e^{2\pi i \alpha} \mathbf{w}\},$$

where $t \in \mathbb{C}^{\times}$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action of \mathbb{Z} generated by the multiplication by $e^{2\pi i \alpha}$. It is known as a *Hopf manifold*.

Theorem 3.3 may be generalised to a wider class of manifolds, namely, to *partial quotients* of $\mathcal{Z}_{\mathcal{K}}$ in the sense of [BP2, §7.5]. Assume there exist a primitive sublattice $N \subset \mathbb{Z}^m$ of rank k such that $\Lambda_{\mathbb{R}} N = 0$. We denote the corresponding k -torus by $T(N) \subset \mathbb{T}^m$.

Proposition 3.5. *The torus $T(N)$ acts freely on $U(\mathcal{K})$.*

Proof. As in the proof of the freeness of the R_{Σ} -action in Theorem 2.2, all \mathbb{T}^m -isotropy subgroups are of the form $(\mathbb{T}, 1)^I$ for $I \in \mathcal{K}$. Since primitive sublattices $(\mathbb{Z}, 0)^I$ and N in \mathbb{Z}^m intersect trivially, the intersection of the corresponding tori $(\mathbb{T}, 1)^I$ and $T(N)$ is also trivial. \square

Let $N_{\mathbb{R}} \subset \mathbb{R}^m$ be the linear subspace generated by N , and let $\rho: \mathbb{R}^m \rightarrow \mathbb{R}^m/N_{\mathbb{R}}$ be the quotient projection. Let $T_{\mathbb{C}}(N) = N \otimes_{\mathbb{Z}} \mathbb{C}^{\times} \cong (\mathbb{C}^{\times})^k$ be the algebraic torus corresponding to N . Assuming that $2\ell = m - n - k$ is even, we have the following generalisation of Construction 3.1.

Construction 3.6. Choose a linear map $\Omega: \mathbb{C}^{\ell} \rightarrow \mathbb{C}^m$ satisfying the following two conditions:

- (a) $\rho \circ \text{Re} \circ \Omega$ is a monomorphism.
- (b) $\Lambda_{\mathbb{R}} \circ \text{Re} \circ \Omega = 0$.

It follows from (a) that the subgroups $T_{\mathbb{C}}(N)$ and $\exp \Omega(\mathbb{C}^{\ell})$ have trivial intersection in $(\mathbb{C}^{\times})^m$, and therefore we may define a subgroup

$$(3.4) \quad C_{\Omega, \Sigma}(N) := T_{\mathbb{C}}(N) \times \exp \Omega(\mathbb{C}^{\ell}) \subset (\mathbb{C}^{\times})^m.$$

Theorem 3.7. *Let Σ and \mathcal{K} be as in Theorem 3.3, and choose N and $C_{\Omega, \Sigma}(N)$ as in Construction 3.6. Then*

- (a) *the holomorphic action of the group $C_{\Psi, \Sigma}(N)$ on $U(\mathcal{K})$ is free and proper, and the quotient $U(\mathcal{K})/C_{\Omega, \Sigma}(N)$ has a structure of a compact complex manifold of complex dimension $m - \ell - k$;*
- (b) *there is a \mathbb{T}^m -equivariant diffeomorphism between $U(\mathcal{K})/C_{\Omega, \Sigma}(N)$ and $\mathcal{Z}_{\mathcal{K}}/T(N)$ defining a complex structure on the quotient $\mathcal{Z}_{\mathcal{K}}/T(N)$, in which $\mathbb{T}^m/T(N)$ acts by holomorphic transformations.*

The proof is similar to that of Theorem 3.3 and is omitted.

Example 3.8. 1. The case $k = 0$ gives back Theorem 3.3.

2. Let $k = 1$ and take N to be the diagonal sublattice of rank one in \mathbb{Z}^m . The condition $\Lambda_{\mathbb{R}}N = 0$ implies that the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ sum up to zero, which can always be achieved by rescaling them (as Σ is a complete fan). As the result, we obtain a complex structure on the quotient of $\mathcal{Z}_{\mathcal{K}}$ by the diagonal subgroup in \mathbb{T}^m , provided that $m - n$ is odd. In the polytopal case $\mathcal{K} = \mathcal{K}_P$ this gives the construction of *LVM manifolds* from [Me] (see also [BM] and the discussion in the next section).

3. Let $k = m - n$. Then $\Lambda_{\mathbb{R}}N = 0$ implies that the whole $\text{Ker } \Lambda_{\mathbb{R}}$ is generated by a primitive lattice. In this case $\ell = 0$ and $C_{\Omega, \Sigma}(N) = T_{\mathbb{C}}(N)$, see (3.4). The quotient $U(\mathcal{K})/T_{\mathbb{C}}(N) = \mathcal{Z}_{\mathcal{K}}/T(N)$ is the *toric variety* corresponding to the rational fan Σ , see Section 5.

4. MOMENT-ANGLE MANIFOLDS FROM POLYTOPES

Normal fans of simple convex polytopes P constitute an important class of complete simplicial fans. The results of the previous section have interesting specifications in the polytopal case. Polytopal moment-angle manifolds \mathcal{Z}_P admit an explicit description as intersections of quadratic hypersurfaces in \mathbb{C}^m . This description originates from [BP2, Constr. 6.8], and explicit quadratic equations for \mathcal{Z}_P were written in [BPR, (3.3)].

Since polytopal moment-angle manifolds \mathcal{Z}_P were shown to be complex in [BM], we find it interesting and important to relate explicitly the quadratic description of \mathcal{Z}_P from [BP2] and [BPR] to the quadratic description of LVM-manifolds from [Me] and [BM]. It helps to understand better how the complex structures on the polytopal moment-angle manifolds \mathcal{Z}_P fit the more general framework of the previous section.

Let P be an n -dimensional convex polytope given as an intersection of m half-spaces in a Euclidean space \mathbb{R}^n with the scalar product $\langle \cdot, \cdot \rangle$:

$$(4.1) \quad P := \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0 \quad \text{for } i = 1, \dots, m \},$$

where $\mathbf{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. We refer to the right hand side of (4.1) as a *presentation* of P by a system of linear inequalities. An inequality $\langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0$ is *redundant* if it can be removed from the presentation without changing P . A presentation of P without redundant inequalities (an *irredundant* presentation) is unique; however for the reasons explained below we allow redundant inequalities and therefore indeterminacy in the presentation of P . We consider the hyperplanes

$$H_i := \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i = 0 \}$$

for $i = 1, \dots, m$, and refer to a presentation of P as *generic* if at most n hyperplanes H_i meet at every point of P . The existence of a generic presentation implies that P is *simple*, that is, there exactly n facets meet at every vertex of P . A generic presentation may contain redundant inequalities, but for every such inequality the intersection of the corresponding hyperplane H_i with P is empty (that is, the inequality is strict for every $\mathbf{x} \in P$).

Construction 4.1. Let A_P be the $m \times n$ matrix of row vectors \mathbf{a}_i and \mathbf{b}_P be the column vector of scalars $b_i \in \mathbb{R}^n$. Consider the affine map

$$i_P: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad i_P(\mathbf{x}) = A_P \mathbf{x} + \mathbf{b}_P.$$

It is monomorphic onto a certain n -dimensional plane in \mathbb{R}^m , and $i_P(P)$ is the intersection of this plane with \mathbb{R}_{\geq}^m .

We define the space \mathcal{Z}_P from the commutative diagram

$$(4.2) \quad \begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \downarrow & & \downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}_{\geq}^m \end{array}$$

where $\mu(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$. The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus \mathbb{T}^m on \mathbb{C}^m . Therefore, \mathbb{T}^m acts on \mathcal{Z}_P with quotient P , and i_Z is a \mathbb{T}^m -equivariant embedding.

Now choose an $(m-n) \times m$ matrix $\Gamma = (\gamma_{jk})$ whose rows form a basis of linear relations between the vectors \mathbf{a}_i , $i = 1, \dots, m$. That is, the corresponding map $\Gamma: \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$ completes the short exact sequence

$$(4.3) \quad 0 \longrightarrow \mathbb{R}^n \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{\Gamma} \mathbb{R}^{m-n} \longrightarrow 0.$$

Then we may write the image of P under i_P by linear equations as

$$(4.4) \quad i_P(P) = \{\mathbf{y} \in \mathbb{R}_{\geq}^m: \Gamma \mathbf{y} = \Gamma \mathbf{b}_P, \text{ for } 1 \leq i \leq m\}.$$

Comparing this with diagram (4.2) we finally obtain that \mathcal{Z}_P embeds into \mathbb{C}^m as the set of common zeros of $m-n$ real quadratic equations

$$(4.5) \quad \sum_{k=1}^m \gamma_{jk} |z_k|^2 = \sum_{k=1}^m \gamma_{jk} b_k, \text{ for } 1 \leq j \leq m-n.$$

The following properties of \mathcal{Z}_P are immediate consequences of its construction.

- Proposition 4.2.** (1) *Given a point $z \in \mathcal{Z}_P$, the k th coordinate of $i_Z(z) \in \mathbb{C}^m$ vanishes if and only if z projects onto a point $\mathbf{x} \in P$ such that $\mathbf{x} \in H_k$.*
 (2) *Adding a redundant inequality to (4.1) results in multiplying \mathcal{Z}_P by a circle.*

Theorem 4.3. *The intersection of quadrics (4.5) defining \mathcal{Z}_P is nondegenerate (transverse) if and only if presentation (4.1) is generic.*

Proof. For simplicity, we do not distinguish between \mathcal{Z}_P and its image $i_Z(\mathcal{Z}_P)$ in \mathbb{C}^m in this proof. Using identification (3.1) we observe that the gradients of the $m-n$ quadratic forms in (4.5) at a point $\mathbf{z} = (x_1, y_1, \dots, x_m, y_m) \in \mathcal{Z}_P$ are

$$(4.6) \quad 2(\gamma_{j1}x_1, \gamma_{j1}y_1, \dots, \gamma_{jm}x_m, \gamma_{jm}y_m), \quad 1 \leq j \leq m-n.$$

These gradients form the rows of the $(m-n) \times 2m$ matrix $2\Gamma\Delta$, where

$$\Delta = \begin{pmatrix} x_1 & y_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & x_m & y_m \end{pmatrix}.$$

Assume that for the chosen point $\mathbf{z} \in \mathcal{Z}_P$ we have $z_{j_1} = \dots = z_{j_k} = 0$, and the other complex coordinates are nonzero. Then the rank of the gradient matrix $2\Gamma\Delta$ at \mathbf{z} is equal to the rank of the $(m-n) \times (m-k)$ matrix Γ' obtained by deleting the columns j_1, \dots, j_k from Γ .

By Proposition 4.2, the point \mathbf{z} projects onto $\mathbf{x} \in H_{j_1} \cap \dots \cap H_{j_k}$.

If presentation (4.1) is generic, then the hyperplanes H_{j_1}, \dots, H_{j_k} meet transversely (in particular, $k \leq n$). It follows that the rank of the $k \times n$ submatrix A' of A_P formed by the rows $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_k}$ is k . This implies that Γ' has rank $m-n$ (see [BPR, Lemma 2.18]), and therefore (4.5) is nondegenerate at \mathbf{z} .

On the other hand, if (4.1) is not generic, then we consider a point $\mathbf{z} \in \mathcal{Z}_P$ which projects to the intersection of $k > n$ hyperplanes H_i . Then for this \mathbf{z} the matrix Γ' has $m - k < m - n$ columns, and therefore its rank is less than $m - n$. It follows that (4.5) is degenerate at \mathbf{z} . \square

Corollary 4.4 ([BP2, Cor. 3.9], [BPR, Lemma 4.2]). *\mathcal{Z}_P is a smooth manifold of dimension $m + n$. Moreover, the embedding $i_z: \mathcal{Z}_P \rightarrow \mathbb{C}^m$ is T^m -equivariantly framed by any choice of matrix Γ in (4.3).*

Assume now that (4.1) is generic; in particular, P is simple. Set

$$F_i := H_i \cap P = \{\mathbf{x} \in P: \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i = 0\}$$

for $i = 1, \dots, m$. Note that F_i is either empty (if the i th inequality is redundant), or is a facet of P . The *normal fan* Σ_P of P consists of cones spanned by the sets of vectors $\{\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_k}\}$ for which the intersection $F_{j_1} \cap \dots \cap F_{j_k}$ is nonempty. It is a complete simplicial fan in \mathbb{R}^n . We formally add to Σ_P ‘empty 1-dimensional cones’ corresponding to redundant inequalities (or ‘empty facets’ F_i). Then the underlying simplicial complex $\mathcal{K}_P = \mathcal{K}_{\Sigma_P}$ is on the set $[m]$ and has a ghost vertex for every redundant inequality.

Proposition 4.5. *The manifolds $\mathcal{Z}_{\mathcal{K}_P}$ and \mathcal{Z}_P defined by (2.1) and (4.2) are T^m -equivariantly homeomorphic.*

Proof. The idea is to write both $\mathcal{Z}_{\mathcal{K}_P}$ and \mathcal{Z}_P as a certain identification space $P \times \mathbb{T}^m / \sim$, as in [DJ].

We consider the map $\mathbb{R}_{\geq} \times \mathbb{T} \rightarrow \mathbb{C}$ defined by $(y, t) \mapsto yt$. Taking product we obtain a map $\mathbb{R}_{\geq}^m \times \mathbb{T}^m \rightarrow \mathbb{C}^m$. The preimage of a point $\mathbf{z} \in \mathbb{C}^m$ under this map is $\mathbf{y} \times (\mathbb{T}, 1)^{I(\mathbf{z})}$, where $y_i = |z_i|$ for $1 \leq i \leq m$, $I(\mathbf{z}) \subset [m]$ is the set of zero coordinates of \mathbf{z} , and $(\mathbb{T}, 1)^{I(\mathbf{z})} \subset \mathbb{T}^m$ is the coordinate subgroup defined by (2.2). Therefore, \mathbb{C}^m can be identified with the quotient space $\mathbb{R}_{\geq}^m \times \mathbb{T}^m / \sim$, where $(\mathbf{y}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2)$ if $\mathbf{t}_1^{-1} \mathbf{t}_2 \in (\mathbb{T}, 1)^{I(\mathbf{y})}$.

From (4.2) we obtain a similar description of \mathcal{Z}_P as an identification space. Namely, given $p \in P$, set $I_p = \{i \in [m]: p \in F_i\}$ (the set of facets containing p). Then \mathcal{Z}_P is T^m -equivariantly homeomorphic to

$$(4.7) \quad P \times \mathbb{T}^m / \sim \text{ where } (p, \mathbf{t}_1) \sim (p, \mathbf{t}_2) \text{ if } \mathbf{t}_1^{-1} \mathbf{t}_2 \in (\mathbb{T}, 1)^{I_p}.$$

On the other hand, $\mathcal{Z}_{\mathcal{K}_P}$ can be defined from a diagram similar to (4.2):

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}_P} & \longrightarrow & \mathbb{D}^m \\ \downarrow & & \downarrow \mu \\ \text{cc}(\mathcal{K}_P) & \longrightarrow & \mathbb{I}^m \end{array}$$

We have $\mathbb{D}^m \cong \mathbb{I}^m \times \mathbb{T}^m / \sim$ by restriction of the corresponding construction for \mathbb{C}^m . Under the identification of P with $\text{cc}(\mathcal{K}_P) \subset \mathbb{I}^m$ a point $p \in P$ is mapped to a point in \mathbb{I}^m whose set of zero coordinates is exactly I_p (see the details in [BP2, §4.2]). Therefore, $\mathcal{Z}_{\mathcal{K}_P}$ can be also identified with (4.7). \square

Remark. It is easy to observe that $\mathcal{Z}_P \subset U(\mathcal{K}_P)$. An argument similar to that outlined in [BM, Lemma 0.8] can be used to show that every orbit of the free action of R_{Σ_P} intersects \mathcal{Z}_P transversely at a single point (see also [Gu, Appendix 1]). Therefore, in the polytopal case the intersection of quadrics (4.5) can be used instead of the moment-angle complex $\mathcal{Z}_{\mathcal{K}_P}$ as a transverse set to the orbits of the

R_{Σ_P} -action. This also implies that the smooth structure on $\mathcal{Z}_{\mathcal{K}_P}$ defined by Theorem 2.2 is equivalent to the smooth structure on the intersection of quadrics (4.5). Another way to see that these two smooth structures coincide is to identify both with the smooth structure on \mathcal{Z}_P from [BP2, Lemma 6.2].

Corollary 4.6. *The \mathbb{T}^m -equivariant homeomorphism type of the manifold \mathcal{Z}_P depends only on the combinatorial type of the polytope P .*

We therefore refer to either $\mathcal{Z}_{\mathcal{K}_P}$ or \mathcal{Z}_P as the *moment-angle manifold corresponding to simple polytope P* , or shortly *polytopal moment-angle manifold*.

Corollary 4.7. *The moment-angle manifold \mathcal{Z}_P admits a complex structure as the quotient $U(\mathcal{K}_P)/C_{\Psi, \Sigma_P}$ if $\dim \mathcal{Z}_P = m+n$ is even. If $m+n$ is odd, then the product $\mathcal{Z}_P \times S^1$ admits such a structure.*

Proof. We identify \mathcal{Z}_P with $\mathcal{Z}_{\mathcal{K}_P}$ and equip the latter with a complex analytic structure of the quotient $U(\mathcal{K}_P)/C_{\Psi, \Sigma_P}$ using Theorem 3.3, provided that $m+n$ is even. If $m+n$ is odd, we add one redundant inequality to (4.1) (the simplest one is $1 \geq 0$, with $\mathbf{a}_i = 0$ and $b_i = 1$); then m increases by one, n does not change, and \mathcal{Z}_P multiplies by a circle. \square

The existence of complex structures on intersections of quadrics similar to (4.5) was established in [BM] as a consequence of the construction of LVM-manifolds in [LV] and [Me]. Namely, in [BM] there were considered transverse intersections of *homogeneous* quadrics in \mathbb{C}^m with a unit sphere:

$$(4.8) \quad \mathcal{L} := \left\{ \mathbf{z} \in \mathbb{C}^m : \begin{array}{l} \sum_{k=1}^m d_{jk} |z_k|^2 = 0, \quad d_{jk} \in \mathbb{R}, \quad 1 \leq j \leq m-n-1, \\ \sum_{k=1}^m |z_k|^2 = 1. \end{array} \right\}$$

The transversality condition (guaranteeing that (4.8) is a smooth manifold of dimension $m+n$) translates via the *Gale transform* to the condition that the quotient of \mathcal{L} by the standard action of \mathbb{T}^m is an n -dimensional simple convex polytope (see [BM, Lemma 0.12]; compare Theorem 4.3 above). Such \mathcal{L} were called *links* in [BM].

The underlying smooth manifold \mathcal{N} of an *LVM manifold* is obtained by dropping the last equation in (4.8) and considering the intersection of the remaining $m-n-1$ homogeneous quadrics in the projective space $\mathbb{C}P^{m-1}$. The complex structure comes from identifying this intersection of quadrics with the orbit space of a free action of \mathbb{C}^ℓ on an open subset in $\mathbb{C}P^{m-1}$. By the construction, \mathcal{N} is the quotient of \mathcal{L} by a free action of a circle. On the other hand, according to [BM, Th. 12.2], every link \mathcal{L} admits a complex structure as an LVM-manifold if its dimension $m+n$ is even; otherwise $\mathcal{L} \times S^1$ admits such a structure.

To relate (4.5) to (4.8) we note the following. Assume for simplicity that all redundant inequalities in (4.5) are of the form $1 \geq 0$. The rows of the coefficient matrix $\Gamma = (\gamma_{jk})$ of (4.5) form a basis in the space of linear relations between the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ in (4.1). Since these vectors are determined only up to a positive multiple, we may assume that $|\mathbf{a}_i| = 1$ if the i th inequality is irredundant. Since P is a convex polytope, one of the relations between the \mathbf{a}_i 's is $\sum_{i=1}^m (\text{vol } F_i) \mathbf{a}_i = \mathbf{0}$, where $\text{vol } F_i \geq 0$ is the volume of the facet F_i . By rescaling the vectors \mathbf{a}_i again we may achieve that $\sum_{i=1}^m \mathbf{a}_i = \mathbf{0}$ (this still leaves one scaling parameter free), and choose the coefficients of this relation as the last row of Γ , that is, $\gamma_{m-n,k} = 1$ for $1 \leq k \leq m$. Then the last of the quadrics in (4.5) takes the form $\sum_{k=1}^m |z_k|^2 = \sum_{k=1}^m b_k$. Summing up all m inequalities of (4.1), using the fact that $\sum_{i=1}^m \mathbf{a}_i = \mathbf{0}$

and noting that at least one of the inequalities is strict for some $\mathbf{x} \in \mathbb{R}^n$, we obtain $\sum_{k=1}^m b_k > 0$. Using up the last free scaling parameter we achieve that $\sum_{k=1}^m b_k = 1$. Then the last quadric in (4.5) becomes $\sum_{k=1}^m |z_k|^2 = 1$, which coincides with the last equation in (4.8). Subtracting this equation with an appropriate coefficient from the other equations in (4.5) we finally transform (4.5) to (4.8).

Since the intersection of quadrics (4.5) can be identified with a link (4.8), the above cited result [BM, Th. 12.2] can be used to equip \mathcal{Z}_P with a complex structure. However, the method of Corollary 4.7 is somewhat more direct; it does not require the passage to projectivisations and LVM-manifolds.

5. HOLOMORPHIC BUNDLES OVER TORIC VARIETIES AND HODGE NUMBERS

A *toric variety* is a normal algebraic variety X on which an algebraic torus $(\mathbb{C}^\times)^n$ acts with a dense orbit. As is well known in algebraic geometry, toric varieties are classified by rational fans [Da]. Under this correspondence, complete fans give rise to complete varieties (compact in the usual topology), normal fans of polytopes to projective varieties, regular fans to nonsingular varieties, and simplicial fans to varieties with mild (orbifold-type) singularities.

A construction due to several authors, which is now often referred to as the ‘Cox construction’ [Co], identifies a toric variety X_Σ corresponding to a rational simplicial fan Σ in \mathbb{R}^n with the (geometric) quotient of $U(\mathcal{K}_\Sigma)$ by the action of a certain $(m - n)$ -dimensional algebraic subgroup G_Σ in $(\mathbb{C}^\times)^m$. (General toric varieties also arise as *categorical* quotients, but we shall not need this here.) The Cox construction may be also regarded as an algebraic version of the construction of Hamiltonian toric manifolds via *symplectic reduction* [Gu].

In the case of rational simplicial polytopal fans Σ_P a construction of [MV] identifies the corresponding projective toric variety X_P as the base of a holomorphic principal *Seifert fibration*, whose total space is the moment-angle manifold \mathcal{Z}_P equipped with a complex structure of an LVM-manifold, and fibre is a compact complex torus of complex dimension $\ell = \frac{m-n}{2}$. (Seifert fibrations are generalisations of holomorphic fibre bundles to the case when the base is an orbifold.) If X_P is a nonsingular projective toric variety, then there is a holomorphic free action of a complex ℓ -dimensional torus $T_{\mathbb{C}}^{2\ell}$ on \mathcal{Z}_P with quotient X_P .

Here we generalise the construction of [MV] to rational complete simplicial fans (not necessarily polytopal). By an application of the Borel spectral sequence to the resulting holomorphic principal fibre bundle $\mathcal{Z}_\Sigma \rightarrow X_\Sigma$ we derive some information about the Hodge numbers of complex structures on moment-angle manifolds.

In this section we assume that the fan Σ is complete, simplicial and rational. We choose as $\mathbf{a}_1, \dots, \mathbf{a}_m$ the primitive integral generators of the 1-dimensional cones.

Construction 5.1 (‘Cox construction’). Let $\Lambda_{\mathbb{C}}: \mathbb{C}^m \rightarrow \mathbb{C}^n$ be the complexification of map (2.4), and consider its complex exponential:

$$\exp \Lambda_{\mathbb{C}}: (\mathbb{C}^\times)^m \rightarrow (\mathbb{C}^\times)^n,$$

$$(z_1, \dots, z_m) \mapsto \left(\prod_{i=1}^m z_i^{a_{i1}}, \dots, \prod_{i=1}^m z_i^{a_{in}} \right)$$

(here we use that the fan is rational and the vectors $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$ are integral; otherwise the map above is not defined). Set $G_\Sigma := \text{Ker}(\exp \Lambda_{\mathbb{C}})$. This is an $(m - n)$ -dimensional algebraic subgroup in $(\mathbb{C}^\times)^m$, hence, it is isomorphic to a product of

$(\mathbb{C}^\times)^{m-n}$ and a finite group, where the finite group is trivial if the fan is regular. The group G_Σ acts almost freely (with finite isotropy subgroups) on the open set $U(\mathcal{K}_\Sigma)$; moreover, this action is free if Σ is a regular fan. This is proved in the same way as the freeness of the action of R_Σ on $U(\mathcal{K}_\Sigma)$ in Theorem 2.2 (a).

The *toric variety* associated with the fan Σ is the quotient $X_\Sigma := U(\mathcal{K}_\Sigma)/G_\Sigma$. It is a complex algebraic variety of dimension n . The variety X_Σ is nonsingular whenever Σ is regular; otherwise it has orbifold-type singularities (i.e. X_Σ is locally isomorphic to a quotient of \mathbb{C}^n by a finite group). The quotient algebraic torus $(\mathbb{C}^\times)^m/G_\Sigma \cong (\mathbb{C}^\times)^n$ acts on X_Σ with a dense orbit.

The variety X_Σ is projective if and only if Σ is the normal fan of a convex polytope P ; in this case we shall denote the variety by X_P .

Assume that $m - n$ is even by adding an empty 1-cone to Σ if necessary, and set $\ell := \frac{m-n}{2}$. We observe that for any choice of Ψ in Construction 3.1 the subgroup $C_{\Psi,\Sigma}$ lies in G_Σ as an ℓ -dimensional complex subgroup. This follows from the fact that, since $\Lambda_{\mathbb{C}}$ is the complexification of a real map, condition (b) of Construction 3.1 is equivalent to $\Lambda_{\mathbb{C}}\Psi = 0$, which implies that $\exp \Lambda_{\mathbb{C}}(C_{\Psi,\Sigma})$ is trivial.

Proposition 5.2.

- (a) *The toric variety X_Σ is identified, as a topological space, with the quotient of $\mathcal{Z}_{\mathcal{K}_\Sigma}$ by the holomorphic action of $G_\Sigma/C_{\Psi,\Sigma}$.*
- (b) *If the fan Σ is regular, then X_Σ is the base of a holomorphic principal bundle with total space $\mathcal{Z}_{\mathcal{K}_\Sigma}$ and fibre the complex torus $G_\Sigma/C_{\Psi,\Sigma}$ of dimension ℓ .*

Proof. To prove (a) we just observe that

$$X_\Sigma = U(\mathcal{K}_\Sigma)/G_\Sigma = (U(\mathcal{K}_\Sigma)/C_{\Psi,\Sigma})/(G_\Sigma/C_{\Psi,\Sigma}) \cong \mathcal{Z}_{\mathcal{K}_\Sigma}/(G_\Sigma/C_{\Psi,\Sigma}),$$

where we used Theorem 3.3. If Σ is regular, then $G_\Sigma \cong (\mathbb{C}^\times)^{m-n}$ and $G_\Sigma/C_{\Psi,\Sigma}$ is a compact complex ℓ -torus by Example 3.2. The action of G_Σ on $U(\mathcal{K}_\Sigma)$ is holomorphic and free in this case, and the same is true for the action of $G_\Sigma/C_{\Psi,\Sigma}$ on $\mathcal{Z}_{\mathcal{K}_\Sigma}$. A holomorphic free actions of the torus $G_\Sigma/C_{\Psi,\Sigma}$ gives rise to a principal bundle, which finishes the proof of (b). \square

Remark. Like in the projective situation of [MV], for singular varieties X_Σ the quotient projection $\mathcal{Z}_{\mathcal{K}_\Sigma} \rightarrow X_\Sigma$ of Proposition 5.2 (a) is a holomorphic principal *Seifert bundle* for an appropriate orbifold structure on X_Σ .

Let M be a complex n -dimensional manifold. The space $\Omega_{\mathbb{C}}^*(M)$ of complex differential forms on M decomposes into a direct sum of the subspaces of (p, q) -forms, $\Omega_{\mathbb{C}}^*(M) = \bigoplus_{0 \leq p, q \leq n} \Omega^{p,q}(M)$, and there is the *Dolbeault differential* $\bar{\partial}: \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$. The dimensions $h^{p,q}(M)$ of the Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}(M)$ are known as the *Hodge numbers* of M . They are important invariants of the complex structure of M .

The Dolbeault cohomology of a compact complex ℓ -dimensional torus $T_{\mathbb{C}}^{2\ell}$ is isomorphic to an exterior algebra on 2ℓ generators:

$$(5.1) \quad H_{\bar{\partial}}^{*,*}(T_{\mathbb{C}}^{2\ell}) \cong \Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell],$$

where $\xi_1, \dots, \xi_\ell \in H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell})$ are the classes of basis holomorphic 1-forms, and $\eta_1, \dots, \eta_\ell \in H_{\bar{\partial}}^{0,1}(T_{\mathbb{C}}^{2\ell})$ are the classes of basis antiholomorphic 1-forms. In particular, the Hodge numbers are given by $h^{p,q}(T_{\mathbb{C}}^{2\ell}) = \binom{\ell}{p} \binom{\ell}{q}$.

The de Rham cohomology of a complete nonsingular toric variety X_Σ admits a Hodge decomposition with nonzero terms in bidegrees (p, p) only [Da, §12]. This together with the cohomology calculation due to Danilov–Jurkiewicz [Da, §10] gives the following description of the Dolbeault cohomology:

$$(5.2) \quad H_{\bar{\partial}}^{*,*}(X_\Sigma) \cong \mathbb{C}[v_1, \dots, v_m]/(\mathcal{I}_{\mathcal{K}_\Sigma} + \mathcal{J}_\Sigma),$$

where $v_i \in H_{\bar{\partial}}^{1,1}(X_\Sigma)$, the ideal $\mathcal{I}_{\mathcal{K}_\Sigma}$ is generated by monomials $v_{i_1} \cdots v_{i_k}$ for which $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}$ do not span a cone of Σ (the *Stanley–Reisner ideal*), and \mathcal{J}_Σ is generated by the linear combinations $\sum_{k=1}^m a_{kj} v_k$ for $1 \leq j \leq n$, where a_{kj} denotes the j th coordinate of \mathbf{a}_k . We have $h^{p,p}(X_\Sigma) = h_p$, where (h_0, h_1, \dots, h_n) is the h -vector of \mathcal{K}_Σ [BP2, §2.1], and $h^{p,q}(X_\Sigma) = 0$ for $p \neq q$.

Theorem 5.4. *Let Σ be a complete rational nonsingular fan in \mathbb{R}^n , with m one-dimensional cones (some of which may be empty), and $m + n$ even. Let $\mathcal{Z}_\mathcal{K} = \mathcal{Z}_{\mathcal{K}_\Sigma}$ be the moment-angle manifold with the complex structure defined by a choice of subgroup $C_{\Psi, \Sigma}$, see (3.3). Then the Dolbeault cohomology algebra $H_{\bar{\partial}}^{*,*}(\mathcal{Z}_\mathcal{K})$ is isomorphic to the cohomology of differential bigraded algebra*

$$(5.3) \quad [\Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell] \otimes H_{\bar{\partial}}^{*,*}(X_\Sigma), d]$$

whose bigrading is defined by (5.1) and (5.2), and differential d of bidegree $(0, 1)$ is defined on the generators as

$$dv_i = d\eta_j = 0, \quad d\xi_j = c(\xi_j), \quad 1 \leq i \leq m, \quad 1 \leq j \leq \ell,$$

where $c: H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell}) \rightarrow H^2(X_\Sigma, \mathbb{C}) = H_{\bar{\partial}}^{1,1}(X_\Sigma)$ is the first Chern class map of the torus principal bundle $\mathcal{Z}_\mathcal{K} \rightarrow X_\Sigma$.

Proof. Let $(B, \delta) = \mathcal{M}(X_\Sigma)$ be the minimal Dolbeault model [FOT] of X_Σ . Consider the differential algebra

$$\begin{aligned} & [\Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell] \otimes B, d] \quad \text{with} \\ & d|_B = \delta, \quad d(\xi_i) = c(\xi_i) \in B^{1,1} \cong H_{\bar{\partial}}^{1,1}(X_\Sigma), \quad d(\eta_i) = 0. \end{aligned}$$

By [FOT, Cor. 4.66], it gives a model for the Dolbeault cohomology ring of the total space $\mathcal{Z}_\mathcal{K}$ of the principal $T_{\mathbb{C}}^{2\ell}$ -bundle $\mathcal{Z}_\mathcal{K} \rightarrow X_\Sigma$, provided that X_Σ is strictly formal in the sense of [FOT, Def. 4.58].

According to [NT, Th. 8], a compact manifold satisfying the $\partial\bar{\partial}$ -lemma is strictly formal, in particular any complete toric variety is strictly formal. The ordinary formality implies the existence of a quasi-isomorphism $\varphi_B: B \rightarrow H_{\bar{\partial}}^{*,*}(X_\Sigma)$, which extends to a quasi-isomorphism

$$\text{id} \otimes \varphi_B: [\Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell] \otimes B, d] \rightarrow [\Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell] \otimes H_{\bar{\partial}}^{*,*}(X_\Sigma), d]$$

by [FHT, Lemma 14.2]. Thus, the differential algebra $[\Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell] \otimes H_{\bar{\partial}}^{*,*}(X_\Sigma), d]$ provides a model for the Dolbeault cohomology of $\mathcal{Z}_\mathcal{K}$, as claimed. \square

The map c can be described explicitly in terms of the matrix Ψ defining the complex structure on $\mathcal{Z}_\mathcal{K}$. We shall also need the real $(m - n) \times m$ matrix $\Gamma = (\gamma_{jk})$ such that $(\gamma_{j1}, \dots, \gamma_{jm})$, $1 \leq j \leq m - n$, is any basis in the space of linear relations between the generators $\mathbf{a}_1, \dots, \mathbf{a}_m$ of the one-dimensional cones of Σ . (This matrix was already considered in Section 4 for the polytopal case.)

Lemma 5.5. *We have*

$$c(\xi_j) = \mu_{j1}v_1 + \dots + \mu_{jm}v_m, \quad 1 \leq j \leq \ell,$$

where $M = (\mu_{jk})$ is an $\ell \times m$ matrix satisfying the two conditions:

- (a) $\Gamma M^t: \mathbb{C}^\ell \rightarrow \mathbb{C}^{2\ell}$ is a monomorphism;
- (b) $M\Psi = 0$.

Proof. Let $\Lambda_{\mathbb{C}}^t: \mathbb{C}^n \rightarrow \mathbb{C}^m$ be the transpose of the complexified map (2.4). Then have $H^1(T_{\mathbb{C}}^{2\ell}; \mathbb{C}) = \mathbb{C}^m / (\Lambda_{\mathbb{C}}^t(\mathbb{C}^n))$ and $H^2(X_{\Sigma}; \mathbb{C}) = \mathbb{C}^{m-k} / (\Lambda_{\mathbb{C}}^t(\mathbb{C}^n))$, where k is the number of ghost vertices in \mathcal{K} . Map c is given by the projection

$$p: \mathbb{C}^m / (\Lambda_{\mathbb{C}}^t(\mathbb{C}^n)) \longrightarrow \mathbb{C}^{m-k} / (\Lambda_{\mathbb{C}}^t(\mathbb{C}^n))$$

which forgets the coordinates in \mathbb{C}^m corresponding to the ghost vertices. We therefore need to identify the subspace of holomorphic differentials $H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell}) \cong \mathbb{C}^\ell$ inside the space of all 1-forms $H^1(T_{\mathbb{C}}^{2\ell}; \mathbb{C}) \cong \mathbb{C}^{2\ell}$. Since

$$T_{\mathbb{C}}^{2\ell} = G_{\Sigma} / C_{\Psi, \Sigma} = (\text{Ker exp } \Lambda_{\mathbb{C}}) / \exp \Psi(\mathbb{C}^\ell),$$

holomorphic differentials on $T_{\mathbb{C}}^{2\ell}$ correspond to linear functions on $\text{Ker } \Lambda_{\mathbb{C}}$ which are $\Psi(\mathbb{C}^\ell)$ -invariant. Every such linear function is a restriction of a $\Psi(\mathbb{C}^\ell)$ -invariant linear function on \mathbb{C}^m to $\text{Ker } \Lambda_{\mathbb{C}} \subset \mathbb{C}^m$. The kernel of this restriction consists of those functions which are $\Lambda_{\mathbb{C}}^t(\mathbb{C}^n)$ -invariant. Condition (b) says exactly that the rows of M are $\Psi(\mathbb{C}^\ell)$ -invariant linear functions. Condition (a) says that the rows of M constitute a basis in the complement to the subspace of $\Lambda_{\mathbb{C}}^t(\mathbb{C}^n)$ -invariant functions in the space of $\Psi(\mathbb{C}^\ell)$ -invariant functions. \square

It is interesting to compare Theorem 5.4 with the following description of the de Rham cohomology of $\mathcal{Z}_{\mathcal{K}}$.

Theorem 5.7 ([BP2, Th. 7.36]). *Let $\mathcal{Z}_{\mathcal{K}}$ be as in Theorem 5.4. Then its de Rham cohomology algebra is isomorphic to the cohomology of the differential graded algebra*

$$[\Lambda[u_1, \dots, u_{m-n}] \otimes H^*(X_{\Sigma}), d],$$

with $\deg u_j = 1$, $\deg v_i = 2$, and differential d defined on the generators as

$$dv_i = 0, \quad du_j = \gamma_{j1}v_1 + \dots + \gamma_{jm}v_m, \quad 1 \leq j \leq m-n.$$

This follows from the more general result [BP2, Th. 7.7] describing the cohomology of $\mathcal{Z}_{\mathcal{K}}$. Like Theorem 5.4, the theorem above may be interpreted as a collapse result for a spectral sequence, this time the Leray–Serre spectral sequence of the principal T^{m-n} -bundle $\mathcal{Z}_{\mathcal{K}} \rightarrow X_{\Sigma}$. For more information about the ordinary cohomology of $\mathcal{Z}_{\mathcal{K}}$ see [BP2, Ch. 7] and [Pa, §4]. There is also a bigrading in the ordinary cohomology of $\mathcal{Z}_{\mathcal{K}}$, which is different from the bigrading in the Dolbeault cohomology. These two bigradings may be merged in the Dolbeault cohomology, providing a four-graded structure.

There are two classical spectral sequences for the Dolbeault cohomology. First, the *Borel spectral sequence* [Bor] of a holomorphic fibre bundle $E \rightarrow B$ with a compact Kähler fibre F , whose E_2 term is $H_{\bar{\partial}}(B) \otimes H_{\bar{\partial}}(F)$ and which converges to $H_{\bar{\partial}}(E)$. Second, the *Frölicher spectral sequence* [GH, §3.5], whose E_1 term is the Dolbeault cohomology of a complex manifold M and which converges to the de Rham cohomology of M . Theorems 5.4 and 5.7 can be interpreted as collapse results for these spectral sequences:

Corollary 5.8.

- (a) *The Borel spectral sequence of the holomorphic principal bundle $\mathcal{Z}_\mathcal{K} \rightarrow X_\Sigma$ collapses at the E_3 term, i.e. $E_3 = E_\infty$;*
- (b) *the Frölicher spectral sequence of $\mathcal{Z}_\mathcal{K}$ collapses at the E_2 term: $E_2 = E_\infty$.*

Proof. To prove (a) we just observe that the differential algebra given by (5.3) is the E_2 term of the Dolbeault spectral sequence, and its cohomology is the E_3 term.

Let us prove (b). By comparing the Dolbeault and de Rham cohomology algebras of $\mathcal{Z}_\mathcal{K}$ given by Theorems 5.4 and 5.7 we observe that the elements $\eta_1, \dots, \eta_\ell \in E_1^{0,1}$ cannot survive in the E_∞ term of the Frölicher spectral sequence. The only possible nontrivial differential on these elements is $d_1: E_1^{0,1} \rightarrow E_1^{1,1}$. By Theorem 5.7, the cohomology algebra of $[E_1, d_1]$ is exactly the de Rham cohomology of $\mathcal{Z}_\mathcal{K}$. \square

Lemma 5.9. *Let k be the number of ghost vertices in \mathcal{K} . Then*

$$k - \ell \leq \dim_{\mathbb{C}} \text{Ker}(c: H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell}) \rightarrow H_{\bar{\partial}}^{1,1}(X_\Sigma)) \leq \left\lfloor \frac{k}{2} \right\rfloor.$$

In particular, if $k \leq 1$ then the map c is monic.

Proof. The map c is given by the composite map in the top line of the diagram

$$\begin{array}{ccccc} \mathbb{C}^\ell & \xrightarrow{\Theta} & \mathbb{C}^m / (\Lambda_{\mathbb{C}}^t(\mathbb{C}^n)) & \xrightarrow{p} & \mathbb{C}^{m-k} / (\Lambda_{\mathbb{C}}^t(\mathbb{C}^n)) \\ \downarrow \cong & & \downarrow \text{Re} & & \downarrow \text{Re} \\ \mathbb{R}^{m-n} & \xlongequal{\quad} & \mathbb{R}^{m-n} & \xrightarrow{p'} & \mathbb{R}^{m-n-k} \end{array},$$

where Θ is an inclusion of a complex subspace described in the proof of Lemma 5.5 and p, p' are the projections forgetting the ghost vertices. The left vertical arrow is an (\mathbb{R} -linear) isomorphism, as it has the form $H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell}) \rightarrow H^1(T_{\mathbb{C}}^{2\ell}, \mathbb{C}) \rightarrow H^1(T_{\mathbb{C}}^{2\ell}, \mathbb{R})$, and any real-valued function on the lattice determining the torus $T_{\mathbb{C}}^{2\ell}$ is the real part of the restriction of a \mathbb{C} -linear function to the same lattice.

Since the diagram above is commutative, the kernel of $c = p \circ \Theta$ has the real dimension at most k , which implies the required upper bound on its complex dimension. For the lower bound, we have $\dim_{\mathbb{C}} \text{Ker } c \geq \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell}) - \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,1}(X_\Sigma) = \ell - (2\ell - k) = k - \ell$. \square

Theorem 5.10. *Let $\mathcal{Z}_\mathcal{K}$ be as in Theorem 5.4, and let k be the number of ghost vertices in \mathcal{K} . Then the Hodge numbers $h^{p,q} = h^{p,q}(\mathcal{Z}_\mathcal{K})$ satisfy*

- (a) $\binom{k-\ell}{p} \leq h^{p,0} \leq \binom{\lfloor k/2 \rfloor}{p}$ for $p \geq 0$;
- (b) $h^{0,q} = \binom{\ell}{q}$ for $q \geq 0$;
- (c) $h^{1,q} = (\ell - k) \binom{\ell}{q-1} + h^{1,0} \binom{\ell+1}{q}$ for $q \geq 1$;
- (d) $\frac{\ell(3\ell+1)}{2} - h_2(\mathcal{K}) - \ell k + (\ell+1)h^{2,0} \leq h^{2,1} \leq \frac{\ell(3\ell+1)}{2} - \ell k + (\ell+1)h^{2,0}$.

Proof. Let $A^{p,q}$ denote the bidegree (p, q) component of the differential algebra from Theorem 5.4, and let $Z^{p,q} \subset A^{p,q}$ denote the subspace of d -cocycles. Then $d^{1,0}: A^{1,0} \rightarrow Z^{1,1}$ coincides with the map c , and the required bounds for $h^{1,0} = \dim \text{Ker } d^{1,0}$ are already established in Lemma 5.9. Since $h^{p,0} = \dim \text{Ker } d^{p,0}$ is the p th exterior power of the space $\text{Ker } d^{1,0}$, statement (a) follows.

The differential is trivial on $A^{0,q}$, hence $h^{0,q} = \dim A^{0,q}$, proving (b).

The space $Z^{1,1}$ is spanned by the v_i and $\xi_i \eta_j$ with $\xi_i \in \text{Ker } d^{1,0}$. Hence, $\dim Z^{1,1} = 2\ell - k + h^{1,0}\ell$. Also, $\dim d(A^{1,0}) = \ell - h^{1,0}$, hence $h^{1,1} = \ell - k + h^{1,0}(\ell+1)$.

Similarly, $\dim Z^{1,q} = (2\ell - k) \binom{\ell}{q-1} + h^{1,0} \binom{\ell}{q}$ (spanned by the elements $v_i \eta_{j_1} \cdots \eta_{j_{q-1}}$ and $\xi_i \eta_{j_1} \cdots \eta_{j_q}$ where $\xi_i \in \text{Ker } d^{1,0}$, $j_1 < \dots < j_q$), and $d: A^{1,q-1} \rightarrow Z^{1,q}$ hits a subspace of dimension $(\ell - h^{1,0}) \binom{\ell}{q-1}$. This proves (c).

We have $A^{2,1} = V \oplus W$, where V is spanned by the monomials $\xi_i v_j$ and W by the monomials $\xi_i \xi_j \eta_k$. Therefore,

$$(5.4) \quad h^{2,1} = \dim V - \dim dV + \dim W - \dim dW - \dim dA^{2,0}.$$

Now $\dim V = \ell(2\ell - k)$, $0 \leq \dim dV \leq h_2(\mathcal{K})$ (since $dV \subset H_{\bar{\partial}}^{2,2}(X_{\Sigma})$), $\dim W - \dim dW = \dim \text{Ker } d|_W = \ell h^{2,0}$, and $\dim dA^{2,0} = \binom{\ell}{2} - h^{2,0}$. Plugging these values into (5.4) we obtain the inequalities of (d). \square

Remark. At most one ghost vertex is required to make $\dim \mathcal{Z}_{\mathcal{K}} = m + n$ even. Note that $k \leq 1$ implies $h^{p,0}(\mathcal{Z}_{\mathcal{K}}) = 0$, so that $\mathcal{Z}_{\mathcal{K}}$ does not have holomorphic forms of any degree in this case. If $\mathcal{Z}_{\mathcal{K}}$ is a torus, then $m = k = 2\ell$, and $h^{1,0}(\mathcal{Z}_{\mathcal{K}}) = h^{0,1}(\mathcal{Z}_{\mathcal{K}}) = \ell$. Otherwise Theorem 5.10 implies that $h^{1,0}(\mathcal{Z}_{\mathcal{K}}) < h^{0,1}(\mathcal{Z}_{\mathcal{K}})$, and therefore $\mathcal{Z}_{\mathcal{K}}$ is not Kähler (in the polytopal case this was observed in [Me, Th. 3]).

Example 5.11. Let $\mathcal{Z}_P \cong S^1 \times S^{2n+1}$ be a Hopf manifold of Example 3.4. The corresponding fan is the normal fan Σ_P of the standard n -dimensional simplex P with one redundant inequality. We have $X_P = \mathbb{C}P^n$, and (5.2) describes its cohomology as the quotient of $\mathbb{C}[v_1, \dots, v_{n+2}]$ by the two ideals: \mathcal{I} generated by v_1 and $v_2 \cdots v_{n+2}$, and \mathcal{J} generated by $v_2 - v_{n+2}, \dots, v_{n+1} - v_{n+2}$. The differential algebra of Theorem 5.4 is therefore given by $[\Lambda[\xi, \eta] \otimes \mathbb{C}[t]/t^{n+1}, d]$, and it is straightforward to check that $dt = d\eta = 0$ and $d\xi = t$ for a proper choice of t . The nontrivial cohomology classes are represented by the cocycles $1, \eta, \xi t^n$ and $\xi \eta t^n$, which gives the following nonzero Hodge numbers of \mathcal{Z}_P : $h^{0,0} = h^{0,1} = h^{n+1,n} = h^{n+1,n+1} = 1$.

Example 5.12 (Calabi–Eckmann manifold). Let $P = \Delta^p \times \Delta^q$ be the product of two standard simplices with $p \leq q$, so that $n = p + q$, $m = n + 2$ and $\ell = 1$. The corresponding toric variety is $X_P = \mathbb{C}P^p \times \mathbb{C}P^q$ and its cohomology ring is isomorphic to $\mathbb{C}[x, y]/(x^{p+1}, y^{q+1})$. We may choose $\Psi = (1, \dots, 1, \alpha, \dots, \alpha)^t$ in Construction 3.1, where the number of units is $p + 1$, the number of α 's is $q + 1$ and $\alpha \notin \mathbb{R}$. This provides $\mathcal{Z}_P \cong S^{2p+1} \times S^{2q+1}$ with a structure of a complex manifold of dimension $p + q + 1$, known as a *Calabi–Eckmann manifold*; we denote complex manifolds obtained in this way by $\mathcal{CE}(p, q)$. They are total spaces of holomorphic principal bundles $\mathcal{CE}(p, q) \rightarrow \mathbb{C}P^p \times \mathbb{C}P^q$ with fibre the complex torus $\mathbb{C}/(\mathbb{Z} \oplus \alpha \mathbb{Z})$.

Theorem 5.4 and Lemma 5.5 provide the following description of the Dolbeault cohomology of $\mathcal{CE}(p, q)$:

$$H_{\bar{\partial}}^{*,*}(\mathcal{CE}(p, q)) \cong H[\Lambda[\xi, \eta] \otimes \mathbb{C}[x, y]/(x^{p+1}, y^{q+1}), d],$$

where $dx = dy = d\eta = 0$ and $d\xi = x - y$ for an appropriate choice of x, y . We therefore obtain

$$(5.5) \quad H_{\bar{\partial}}^{*,*}(\mathcal{CE}(p, q)) \cong \Lambda[\omega, \eta] \otimes \mathbb{C}[x]/(x^{p+1}),$$

where $\omega \in H_{\bar{\partial}}^{q+1,q}(\mathcal{CE}(p, q))$ is the cohomology class of the cocycle $\xi \frac{x^{q+1} - y^{q+1}}{x - y}$. This calculation was done in [Bor, §9] using a slightly different argument. We note that Dolbeault cohomology of a Calabi–Eckmann manifold depends only on p, q and does not depend on the complex parameter α (or matrix Ψ).

Example 5.13. Now let $P = \Delta^1 \times \Delta^1 \times \Delta^2 \times \Delta^2$. Then \mathcal{Z}_P has two structures of a product of Calabi–Eckmann manifolds, namely, $\mathcal{CE}(1, 1) \times \mathcal{CE}(2, 2)$ and $\mathcal{CE}(1, 2) \times \mathcal{CE}(1, 2)$. Using isomorphism (5.5) we observe that these two complex manifolds have different Hodge numbers $h^{2,1}$: it is 1 in the first case, and 0 in the second. This shows that the choice of matrix Ψ affects not only the complex structure of \mathcal{Z}_P , but also its Hodge numbers, unlike the previous examples of complex tori, Hopf and Calabi–Eckmann manifolds. Certainly it is not highly surprising from the complex-analytic point of view.

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